

A Constructive Approach to the Infinite Family of π through Trigonometric Products

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Abstract

This paper introduces a family of infinite product formulae for π , developed through an original and foundational approach that blends pedagogy, history, and mathematical discovery. The methods are constructed from scratch, avoiding reliance on memorised angle-division identities or shortcuts. Instead, the derivations unfold step by step, making the complex structure of trigonometric and nested radical expressions accessible to students with only a basic mathematical background. This process not only facilitates genuine understanding but also enables learners to independently reconstruct the logic of the results—promoting a deeper conceptual grasp over rote techniques. Alongside these pedagogical advantages, the paper makes a distinct research contribution: reveals elegant and compact infinite product representations for π , derived through self-contained trigonometric constructions and supported by Dirichlet kernels. These products converge exponentially, with precision controlled by a single parameter; for instance, one specific choice yields 31 correct digits using just one term. Each trigonometric factor is expressible as a nested radical built from repeated halving and sign tracking, producing predictable patterns that enhance both computation and intuition. Framed within the historical evolution of π —from ancient approximations to modern infinite series—this work exemplifies how mathematical insight, when developed from first principles, can simultaneously advance learning and original inquiry.

1. Introduction

The ratio of a circle's circumference to its diameter, (C/D), has long intrigued mathematicians due to its elusive nature, decimal calculation impossibility, and straight-line geometric representation. Denoted by the Greek letter π , this constant has a rich history spanning from the Indus Valley Civilization to modern computing. Despite calculations reaching 100 trillion digits, its true nature remains a mystery. Before digital computers, mathematicians successfully but inconclusively pursued exact values, gradually replacing slow-converging infinite series with rapidly converging formulas to maximise precision.

Chronological milestones reveal this progression of refinement:

- Circa 2560 BC: The Great Pyramid of Giza's dimensions exhibit a ratio close to $\pi \approx 22/7$ though intentional usage by its builders remains unproven.
- Around 2000 BC: A Babylonian tablet indicates π was approximated using the base-60 system as: $\pi = \frac{57}{60} + \frac{36}{60^2} = 3.125$. (See [3])
- Around 2000 BC: The Rhind Papyrus by Ahmes equates a 9-unit diameter circle to an 8-unit side square, translating to: $\pi \left(\frac{9}{2}\right)^2 = 8^2$ or $\pi \approx 3.1605$
- Around 800 BC: The Shulba Sutras addressed the "squaring the circle" problem by setting a square's side to $13/15$ of a circle's diameter, yielding: $\pi r^2 \approx \left(\frac{13}{15}\right)^2 (4r^2)$ or $\pi \approx 3.0044$. (See [1] and [2].)
- Around 499 AD: Aryabhata provided a highly precise fractional verse translating to: $\pi = \frac{62832}{20000} = 3.1416$. (See [3].)

From 2000 BC to 120 BC, ancient mathematicians gave various formulae in their quest of perfection. For the sake of brevity, these are not mentioned here.

Figure 1 illustrates that the perimeter of a regular 6-sided polygon provides a closer approximation to the circumference of a circle than that of a 5-sided polygon, and that the 8-sided

polygon approximates it even better than the 6-sided one. This leads to the conclusion that as the number of sides n of a regular polygon increases and tends to ∞ , its perimeter approaches the circumference of the circle. This was the profound intuition of the great physicist and mathematician Archimedes.

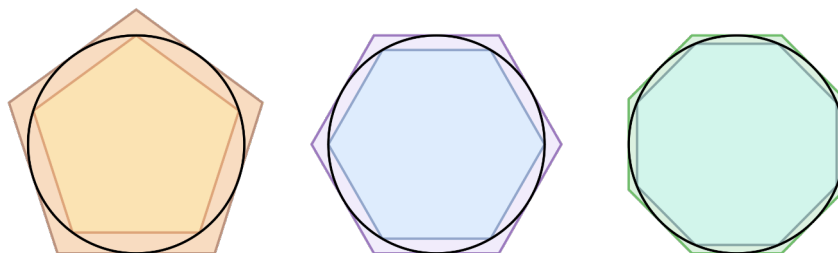


Figure 1 Displaying increasing-sided regular polygons inscribed and circumscribed in a circle- historical method to approximate π

Around 240 BC, Archimedes estimated the value of π by calculating the perimeters of regular polygons inscribed in and circumscribed around a circle, progressively increasing the number of sides of the polygon reaching the inequality

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

The average of $223/71$ and $22/7$ yields the value of $\pi=3.141851$. See [3].

In 1400 CE, Madhava of Sangamagramma developed series expansion of $\tan^{-1}(x)$ as

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

and that helped approximation of π at $x = 1$ (See [3].)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

James Gregory in 1671 and Leibniz in 1673 independently rediscovered the Madhava's formula. Using the series of Madhava and Gregory-Leibniz, John Machin in 1706 gave a fast-converging algorithm. See [4].

$$\frac{\pi}{4} = 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Thereafter, many variants of Machin like formulae were innovated. Sir Issac Newton to make convergence of Madhava series fast gave the formula (See [5] and [6].):

$$\tan^{-1}(x) = \frac{x}{1+x^2} + \left(\frac{2}{3}\right) \frac{x^3}{(1+x^2)^2} + \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) \frac{x^5}{(1+x^2)^3} + \dots$$

Nilakantha Somayaji in the 15th century published a series that had faster convergence rate than that of Madhava series. (See [7].)

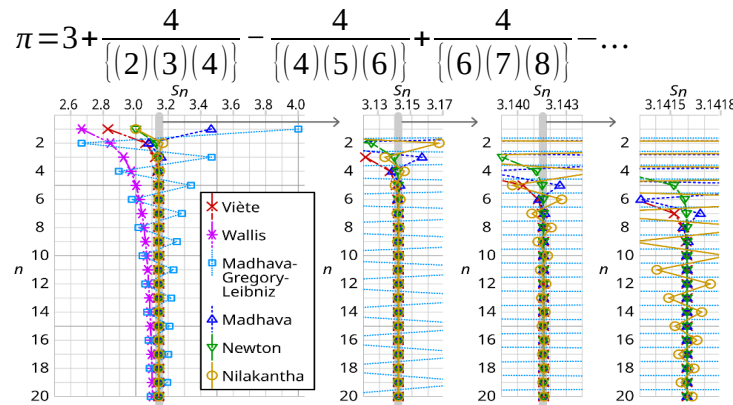


Figure 2 Comparison of convergence of several historical infinite series for π by Cmglee, licensed under [CC-SA 3.0](#)

Figure 2 consists of four subfigures arranged from left to right. The leftmost subfigure, spanning the scale from 2.62 to 4.0, displays the convergence of various series for π —including those of *Viète*, *Wallis*, *Madhava-Gregory-Leibniz*, *Madhava*, *Newton*, and *Neelkantha*—each evaluated up to 20 terms. The next subfigure to the right narrows the scale to 3.13–3.17. Further right, the scale is refined to 3.140–3.143. The rightmost subfigure zooms in further to the scale 3.1415–3.1418, allowing a clearer view of the convergence rates of the series. A visual inspection of these subfigures clearly reveals that *Neelkantha's* series converges the fastest, followed by *Newton's*, then *Madhava's*, then *Viète's*, then *Wallis's*, and finally the *Madhava-Gregory-Leibniz* series.

Coming to the chronological order of development of series for π , Euler in 1735, solved Basel problem and gave the formula

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

and this helped development of Riemann zeta function.

19th Century – Ramanujan and Beyond

In 1914, Srinivasa Ramanujan published several papers developing fast convergence of series summation for π :

$$\frac{1}{\pi} = \left(\frac{2^{\frac{3}{2}}}{9801} \right) \sum_{k=0}^{\infty} \frac{(4k!)(1103+26390k)}{(k!)^4(396)^{4k}},$$

(See [8]). Bill Gosper used this series in 1985 for calculating π up to 17 million digits.

The Chudnovsky brothers proposed formula

$$\frac{1}{\pi} = \left(\frac{10005^{\frac{1}{2}}}{4270934400} \right) \sum_{k=0}^{\infty} \frac{(6k!)(13591409+545140134k)}{[(3k!)(k!)^3(-640320)^{3k}},$$

in 1988 to calculate π up to 1 billion digit and Alexander Yee and Shigeru Kondo used it in 2013 for calculating π up to 10 trillion digits. Emma Haruka Iwao used it in 2022 for 100 trillion digits.

Figure 3 Buffon needle by Inductive load, Commons, licensed [CC BY-SA 3.0](#)

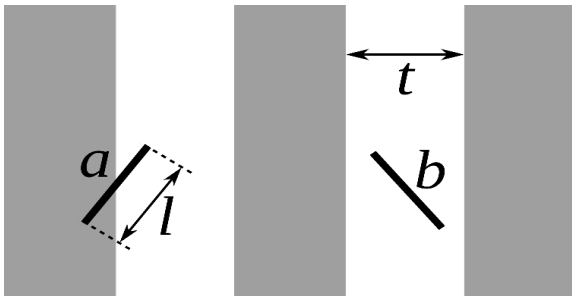
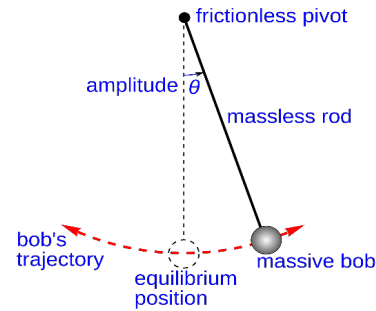


Figure 4 Simple pendulum Wikimedia



Other methods include Monte Carlo, which relies on the results of multiple random trials, and Buffon's needle, which involves dropping a needle of length l a total of n times onto a surface marked with parallel lines spaced t units apart. The needle crosses a line x times, yielding an approximation of π as $2nl/xt$. Notwithstanding the methods stated above, π is approximated by infinite products.

In 1593, Francois Viete first from Europe gave infinite product in nested radicals using the identity,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right),$$

(See [10]). At $x = \pi/4$, on simplification, he could calculate π by formula

$$\frac{2}{\pi} = \left(\frac{2^{\frac{1}{2}}}{2}\right) \left\{ \frac{\left(2 + 2^{\frac{1}{2}}\right)^{\frac{1}{2}}}{2} \right\} \left[\frac{\left\{ 2 + \left(2 + 2^{\frac{1}{2}}\right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}}{2} \right] \dots$$

John Wallis in 1655 gave an infinite product in rational ratios

$$\frac{\pi}{2} = \left\{ \left(\frac{2}{1}\right)\left(\frac{2}{3}\right) \right\} \left\{ \left(\frac{4}{3}\right)\left(\frac{4}{5}\right) \right\} \left\{ \left(\frac{6}{5}\right)\left(\frac{6}{7}\right) \right\} \dots$$

Euler also gave the infinite product form for $\sin(x)/x$ as

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right)$$

and at $x = \pi/2$, the equation takes the form

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right)$$

This is **equivalent** to Wallis' product. However, at $x = \pi/4$, the equation takes the form

$$\frac{2^{\frac{3}{2}}}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{16n^2}\right).$$

Euler also gave infinite product for π

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=2}^{\infty} \frac{1}{n^s},$$

where $\zeta(s)$ is a Riemann zeta function, p denotes prime numbers, n natural numbers, $s > 1$ its power and $|p^{-s}| < 1$. At $s=2$, we can have product form

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)^{-1} = \frac{\pi^2}{6}.$$

With this rich history of π , its applications are enumerated here.

- i. In *Geometry and trigonometry*, it is used for calculating volume, area and length of geometric shapes such as ellipses, spheres, cones, area, circumference, arc length. Barbier theorem, Reuleaux triangle also involve π .
- ii. Angles are measured in radians (1 degree = $\pi/180$ radian) and that involves π .
- iii. Even where the geometry is not involved, π is involved in time period of a common pendulum and eigenvalue.
- iv. It appears in Fourier transforms and Heisenberg uncertainty principle.
- v. The π constant appears in Gaussian function which is the probability density function of the normal distribution with mean μ and standard deviation σ and contains π .
- vi. It appears in topology, Cauchy integral formula.
- vii. The π constant has its role in vector calculus and physics.
- viii. It has its role in Gamma function, Stirling's approximations.
- ix. Riemann zeta function at $s=2$, involves π in formula, $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.
- x. Modular forms, theta functions and Mandelbrot set also involve mathematical constant π .

Purpose of writing this paper is to

- i. Explain geometrically nature and transcendence of π and its impossibility in squaring a circle (Please refer to section 2),
- ii. Devise formula for summation of fixed angle series: $\sum_{m=1}^K \cos\{2m\pi/(2k+1)\} = -1/2$ (Please refer to section 2),
- iii. Devise methods for determining value of $\cos(\pi/n)$ and $\sin(\pi/n)$ (Please refer to sections 5 and 6),
- iv. Discover formulae

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2x}{(2K+1)^n}\right\} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{4x}{(2K+1)^n}\right\} + \dots + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2x}{(2K+1)^n}\right\} \right]$$

where K is a positive integer and

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left(\frac{1}{2K+1} \right) \frac{\sin\left\{\frac{x}{(2K+1)^{n-1}}\right\}}{(2K+1) \sin\left\{\frac{x}{(2K+1)^n}\right\}}.$$

for faster convergence and evaluation of π . Please refer to sections 3, 4, 7 and 8.

2. Nature of π

The nature of π has always been intriguing. In 1761, Johann Heinrich Lambert proved that π is an irrational number. See [9]. Irrationality means that the number cannot be expressed as a ratio of two integers, although it may still be a root of a polynomial equation with rational coefficients.

Lambert considered a continued fraction expansion

$$\tan(x) = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{7 - \dots}}}}$$

based on infinite products and used it to devise a family of infinite formulae for π with faster convergence.

Through a rigorous proof, he demonstrated that if x is a nonzero rational number, then $\tan(x)$ must be irrational, as the continued fraction for $\tan(x)$ progresses infinitely. However, at $x = \pi/4$, since $\tan(x) = 1$, which is rational, Lambert concluded that $\pi/4$ itself must be irrational. If $\pi/4$ were rational, then $\tan(\pi/4)$ would have been irrational, contradicting the known result $\tan(\pi/4) = 1$. Therefore, by contrapositive reasoning, π must be irrational.

While irrational numbers can be roots of algebraic equations, a number that cannot be the root of any polynomial equation with rational coefficients is called transcendental. In 1882, Ferdinand von Lindemann published a paper proving that if a is a nonzero algebraic number, then e^a is transcendental. He applied this to $a = i\pi$, noting that i is algebraic, hence $e^{i\pi} = -1$ (by Euler's identity), which is algebraic. Thus, for this to hold, π must be transcendental. Lindemann's method was later generalised by *Karl Weierstrass*, and the resulting statement is known as the *Lindemann–Weierstrass* theorem.

2.1. Transcendence And Squaring a Circle

The circumference of a circle with radius r is $2\pi r$, and it can only be determined if the exact value of π is known. However, since π is a transcendental number, its value remains perpetually elusive—no matter how hard one tries, its exactitude can never be attained. There is no way to measure the curved circumference of a circle without invoking the transcendental constant π . Under the constraints of Euclidean geometry, this curved length cannot be straightened to form the perimeter of a square. Moreover, the use of mechanical methods, graduated scales, or other instruments are not permitted in classical constructions. Therefore, squaring the circle—or equating its circumference to that of a square—is impossible.

3. Determining Formulae for $\sin(x)/x$

3.1 One-Third Angle Division

For real values of x , identity in complex numbers $e^{ix} = (e^{ix/3})^3$ is expanded

$$e^{ix} = \cos(x) + i \sin(x) = \left\{ \cos\left(\frac{x}{3}\right) + i \sin\left(\frac{x}{3}\right) \right\}^3$$

$$= \cos^3\left(\frac{x}{3}\right) - i \sin^3\left(\frac{x}{3}\right) + 3i \cos^2\left(\frac{x}{3}\right) \sin\left(\frac{x}{3}\right) - 3 \cos\left(\frac{x}{3}\right) \sin^2\left(\frac{x}{3}\right).$$

Comparing imaginary parts, $\sin(x) = -\sin^3(x/3) + 3 \cos^2(x/3) \sin(x/3)$ or

$$\sin(x) = \left\{ 3 - 4 \sin^2\left(\frac{x}{3}\right) \right\} \left\{ \sin\left(\frac{x}{3}\right) \right\}. \quad (3.1)$$

Replacing only $\sin(x/3)$ with $\{3 - 4 \sin^2(x/3)\} \{\sin(x/3)\}$ appearing at extreme right in right hand side (RHS) of Equation (3.1), we get

$$\sin(x) = \left\{ 3 - 4 \sin^2\left(\frac{x}{3}\right) \right\} \left\{ 3 - 4 \sin^2\left(\frac{x}{3^2}\right) \right\} \left\{ \sin\left(\frac{x}{3^2}\right) \right\}. \quad (3.2)$$

Similarly, replacing $\sin(x/3^2)$ with $\{3 - 4 \sin^2(x/3^3)\} \{\sin(x/3^3)\}$ appearing at extreme right in right hand side (RHS) of Equation (3.2) and repeating the process n times, we get

$$\sin(x) = \left\{ 3 - 4 \sin^2\left(\frac{x}{3}\right) \right\} \left\{ 3 - 4 \sin^2\left(\frac{x}{3^2}\right) \right\} \left\{ 3 - 4 \sin^2\left(\frac{x}{3^3}\right) \right\} \dots \left\{ 3 - 4 \sin^2\left(\frac{x}{3^n}\right) \right\} \left\{ \sin\left(\frac{x}{3^n}\right) \right\}.$$

This equation simplifies to

$$\frac{\sin(x)}{\sin\left(\frac{x}{3^n}\right)} = \left\{ 3 - 4 \sin^2\left(\frac{x}{3}\right) \right\} \left\{ 3 - 4 \sin^2\left(\frac{x}{3^2}\right) \right\} \left\{ 3 - 4 \sin^2\left(\frac{x}{3^3}\right) \right\} \dots \left\{ 3 - 4 \sin^2\left(\frac{x}{3^n}\right) \right\}$$

or

$$\frac{\sin(x)}{\sin\left(\frac{x}{3^n}\right)} = 3^n \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3}\right) \right\} \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3^2}\right) \right\} \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3^3}\right) \right\} \dots \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3^n}\right) \right\}$$

When n tends to ∞ , $\sin(x/3^n)$ tends to $x/3^n$ and $\{1 - 4 \sin^2(x/3^n)\}$ tends to 1,

$$(3^n) \frac{\sin(x)}{x} = 3^n \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3}\right) \right\} \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3^2}\right) \right\} \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3^3}\right) \right\} \dots \left\{ 1 - \frac{4}{3} \sin^2\left(\frac{x}{3^n}\right) \right\}$$

or

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left\{ 1 - \left(\frac{4}{3}\right) \sin^2\left(\frac{x}{3^n}\right) \right\}. \quad (3.3)$$

Since $2 \sin^2(x) = 1 - \cos(2x)$, Equation (3.3) is rewritten

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left\{ \frac{1}{3} + \left(\frac{2}{3}\right) \cos\left(\frac{2x}{3^n}\right) \right\}. \quad (3.4)$$

3.2 One-Fifth Angle Division

Following the same procedure as detailed for one third angle division, identity in complex numbers $e^{ix} = (e^{ix/5})^5$ is expanded by Binomial theorem

$$\begin{aligned} \cos(x) + i \sin(x) &= \left\{ \cos\left(\frac{x}{5}\right) + i \sin\left(\frac{x}{5}\right) \right\}^5 \\ &= \cos^5\left(\frac{x}{5}\right) + i 5 \cos^4\left(\frac{x}{5}\right) \sin\left(\frac{x}{5}\right) - 10 \cos^3\left(\frac{x}{5}\right) \sin^2\left(\frac{x}{5}\right) \\ &\quad - i 10 \cos^2\left(\frac{x}{5}\right) \sin^3\left(\frac{x}{5}\right) + 5 \cos\left(\frac{x}{5}\right) \sin^4\left(\frac{x}{5}\right) + i \sin^5\left(\frac{x}{5}\right). \end{aligned}$$

Comparing imaginary parts and converting $\cos(x)$ terms to $\sin(x)$ terms,

$$\sin(x) = \left\{ 16 \sin^4\left(\frac{x}{5}\right) - 20 \sin^2\left(\frac{x}{5}\right) + 5 \right\} \left\{ \sin\left(\frac{x}{5}\right) \right\}. \quad (3.5)$$

Replacing only $\sin(x/5)$ with $\{16 \sin^4(x/5^2) - 20 \sin^2(x/5^2) + 5\}$ appearing at extreme right in right hand side (RHS) of Equation (3.5) and following the same procedure as explained in section (3.1), we obtain

$$\frac{5^n \sin(x)}{x} = 5^n \left\{ \left(\frac{16}{5} \right) \sin^4\left(\frac{x}{5}\right) - 4 \sin^2\left(\frac{x}{5}\right) + 1 \right\} \left\{ \left(\frac{16}{5} \right) \sin^4\left(\frac{x}{5^2}\right) - 4 \sin^2\left(\frac{x}{5^2}\right) + 1 \right\} \left\{ \left(\frac{16}{5} \right) \sin^4\left(\frac{x}{5^3}\right) - 4 \sin^2\left(\frac{x}{5^3}\right) + 1 \right\} \dots \left\{ \left(\frac{16}{5} \right) \sin^4\left(\frac{x}{5^n}\right) - 4 \sin^2\left(\frac{x}{5^n}\right) + 1 \right\}.$$

When n tends to ∞ , this identity can be written as

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ \frac{16}{5} \sin^4\left(\frac{x}{5^n}\right) - 4 \sin^2\left(\frac{x}{5^n}\right) + 1 \right\} \quad (3.6)$$

Using identity $2 \sin^2(x) = 1 - \cos(2x)$, the identity (3.6) simplifies to

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ \frac{1}{5} + \frac{2}{5} \cos\left(\frac{2x}{5^n}\right) + \frac{2}{5} \cos\left(\frac{4x}{5^n}\right) \right\}. \quad (3.7)$$

3.3. One-Seventh Angle Division

Again, following the same procedure as detailed for one third angle division, identity $e^{ix} = (e^{ix/7})^7$ is expanded using Binomial theorem

$$\cos(x) + i \sin(x) = \left\{ \cos\left(\frac{x}{7}\right) + i \sin\left(\frac{x}{7}\right) \right\}^7 = \cos^7\left(\frac{x}{7}\right) + i 7 \cos^6\left(\frac{x}{7}\right) \sin\left(\frac{x}{7}\right) - 21 \cos^5\left(\frac{x}{7}\right) \sin^2\left(\frac{x}{7}\right) - i 35 \cos^4\left(\frac{x}{7}\right) \sin^3\left(\frac{x}{7}\right) + 35 \cos^3\left(\frac{x}{7}\right) \sin^4\left(\frac{x}{7}\right) + i 7 \cos^2\left(\frac{x}{7}\right) \sin^5\left(\frac{x}{7}\right) - 7 \cos\left(\frac{x}{7}\right) \sin^6\left(\frac{x}{7}\right) + \sin^7\left(\frac{x}{7}\right).$$

Comparing imaginary parts and converting $\cos(x)$ terms to $\sin(x)$ terms,

$$\sin(x) = 7 \sin\left(\frac{x}{7}\right) - 56 \sin^3\left(\frac{x}{7}\right) + 112 \sin^5\left(\frac{x}{7}\right) - 64 \sin^7\left(\frac{x}{7}\right).$$

Following the same procedure, we get

$$\frac{7^n \sin(x)}{x} = 7^n \left\{ \left(1 - 8 \sin^2\left(\frac{x}{7}\right) + 16 \sin^4\left(\frac{x}{7}\right) - \left(\frac{64}{7}\right) \sin^6\left(\frac{x}{7}\right) \right) \left\{ 1 - 8 \sin^2\left(\frac{x}{7^2}\right) + 16 \sin^4\left(\frac{x}{7^2}\right) - \left(\frac{64}{7^2}\right) \sin^6\left(\frac{x}{7^2}\right) \right\} \dots \right\}$$

Simplifying

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ 1 - 8 \sin^2\left(\frac{x}{7^n}\right) + 16 \sin^4\left(\frac{x}{7^n}\right) - \frac{64}{7} \sin^6\left(\frac{x}{7^n}\right) \right\}, \quad (3.8)$$

when n tends to ∞ .

Using identities $2 \sin^2(x) = 1 - \cos(2x)$, and $3 \sin(x) - 4 \sin^3(x) = \cos(3x)$, the identity (3.8) simplifies to

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ \frac{1}{7} + \frac{2}{7} \cos\left(\frac{2x}{7^n}\right) + \frac{2}{7} \cos\left(\frac{4x}{7^n}\right) + \frac{2}{7} \cos\left(\frac{6x}{7^n}\right) \right\}. \quad (3.9)$$

3.4. Generalisation by $(2K+1)$ th Angle Division

It is already proved, in one third angle division,

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ \frac{1}{3} + \left(\frac{2}{3}\right) \cos\left(\frac{2x}{3^n}\right) \right\}.$$

in one fifth and one seventh angle division,

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ \frac{1}{5} + \frac{2}{5} \cos\left(\frac{2x}{5^n}\right) + \frac{2}{5} \cos\left(\frac{4x}{5^n}\right) \right\},$$

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ \frac{1}{7} + \frac{2}{7} \cos\left(\frac{2x}{7^n}\right) + \frac{2}{7} \cos\left(\frac{4x}{7^n}\right) + \frac{2}{7} \cos\left(\frac{6x}{7^n}\right) \right\}.$$

By mathematical induction, in one $(2K+1)$ th angle division,

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2x}{(2K+1)^n}\right\} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{4x}{(2K+1)^n}\right\} + \dots + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2Kx}{(2K+1)^n}\right\} \right],$$

where K is a positive integer. This proves the following Lemma 1.

Lemma 1: $\sin(x)/x$ is always expressible in infinite products

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2x}{(2K+1)^n}\right\} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{4x}{(2K+1)^n}\right\} + \dots + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2Kx}{(2K+1)^n}\right\} \right],$$

where K is a positive integer and $-\pi < x < \pi$, $x \neq 0$.

3.5. Fixed Angle Cosine Identities

Here the angle $x = K\pi$ is fixed, where K is a positive or negative integer. Zeros of the function $\sin(x)/x = 0$ are at $x = K\pi$. It does not have a root at $x = 0$, as $\sin(x)/x$ at this value is indeterminate. In general, $\sin(x)/x$, hence $\sin(x)$ is given by Equation (3.10). That means

$$\prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2x}{(2K+1)^n}\right\} + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{4x}{(2K+1)^n}\right\} + \dots + \left(\frac{2}{2K+1}\right) \cos\left\{\frac{2Kx}{(2K+1)^n}\right\} \right] = 0$$

at $x = \pi, 2\pi, 3\pi, \dots$. It can be observed that first multiplicative term $1/3 + (2/3) \cos(2\pi/3)$ is zero at $k=1$, and $n=1$. Second multiplicative term at $x=3\pi$ is $1/3 + (2/3) \cos(6\pi/3^2)$ which again equals to zero. However, it is not equal to zero at $x=\pi$. Similarly, third multiplicative term at $x=9\pi$ is $1/3 + (2/3) \cos(18\pi/3^3)$ which is again zero. Therefore, *first multiplicative term* of general expression of $\sin(x)/x$ at $x=\pi$,

$$\frac{1}{2K+1} + \left(\frac{2}{2K+1}\right) \sum_{m=1}^K \cos\left(\frac{2m\pi}{2K+1}\right) = 0,$$

or

$$\sum_{m=1}^K \cos\left(\frac{2m\pi}{2K+1}\right) = -\frac{1}{2}. \quad (3.11)$$

Example: When $K=9$,

$$\cos\left(\frac{2\pi}{19}\right) + \cos\left(\frac{4\pi}{19}\right) + \cos\left(\frac{6\pi}{19}\right) + \cos\left(\frac{8\pi}{19}\right) + \cos\left(\frac{10\pi}{19}\right) + \cos\left(\frac{12\pi}{19}\right) + \cos\left(\frac{14\pi}{19}\right) + \cos\left(\frac{16\pi}{19}\right) + \cos\left(\frac{18\pi}{19}\right) = -1/2$$

Using the identity, $\cos(\pi - x) = -\cos(x)$, this simplifies to

$$\cos\left(\frac{\pi}{19}\right) - \cos\left(\frac{2\pi}{19}\right) + \cos\left(\frac{3\pi}{19}\right) - \cos\left(\frac{4\pi}{19}\right) + \cos\left(\frac{5\pi}{19}\right) - \cos\left(\frac{6\pi}{19}\right) + \cos\left(\frac{7\pi}{19}\right) - \cos\left(\frac{8\pi}{19}\right) + \cos\left(\frac{9\pi}{19}\right) = \frac{1}{2}.$$

or

$$\sum_{m=1}^9 (-1)^{m+1} \cos\left(\frac{m\pi}{19}\right) = \frac{1}{2}.$$

In general,

$$\sum_{m=1}^K (-1)^{m+1} \cos\left(\frac{m\pi}{2K+1}\right) = \frac{1}{2}. \quad (3.12)$$

4. Determination of Formulae for $\sin(x)/x$ Using Dirichlet Kernel Periodic Function

Dirichlet Kernel is a collection of Periodic Functions

$$D_K(x) = \sum_{m=-K}^K e^{imx} = 1 + \sum_{m=1}^K 2 \cos(mx) = \frac{\sin\left\{\frac{x(2K+1)}{2}\right\}}{\sin\left(\frac{x}{2}\right)}.$$

Replacing x with $2x/(2K+1)^n$,

$$\begin{aligned} \frac{1}{2K+1} D_K\left\{\frac{x}{(2K+1)^n}\right\} &= \frac{1}{2K+1} + \frac{1}{2K+1} \sum_{m=1}^K \left[2 \cos\left\{\frac{2mx}{(2K+1)^n}\right\} \right] \\ &= \left(\frac{1}{2K+1}\right) \frac{\sin\left\{\frac{x}{(2K+1)^{n-1}}\right\}}{\sin\left\{\frac{x}{(2K+1)^n}\right\}} \end{aligned}$$

Therefore, $\sin(x)/x$ in Dirichlet kernel is given by identity

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} D_K\left\{\frac{x}{(2K+1)^n}\right\} \right]$$

(4.1)

and

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\left(\frac{1}{2K+1}\right) \frac{\sin\left\{\frac{x}{(2K+1)^{n-1}}\right\}}{\sin\left\{\frac{x}{(2K+1)^n}\right\}} \right] \quad (4.2)$$

It is observed; these identities yield result on calculation of even one multiplicative term correct up to twenty decimal points provided k is taken extremely large. This proves the following Lemma 2.

Lemma 2: Unnormalised sinc function $\sin(x)/x$ for $x \neq 0$ is always expressible as

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} D_K\left\{\frac{x}{(2K+1)^n}\right\} \right] = \prod_{n=1}^{n=\infty} \left[\left(\frac{1}{2K+1}\right) \frac{\sin\left\{\frac{x}{(2K+1)^{n-1}}\right\}}{\sin\left\{\frac{x}{(2K+1)^n}\right\}} \right],$$

where K is a positive integer, $\sin(x)/x = 1$, when $x \rightarrow 0$, $-\pi < x < \pi$, $x \neq 0$ and

$$D_K(x) = \sum_{m=-K}^K e^{imx} = 1 + \sum_{m=1}^K \{2 \cos(mx)\}$$

is a Dirichlet kernel periodic function. An important question that arises is how to evaluate numerically $\sin\{x/(2K+1)\}$ and $\sin\{x/(2K+1)^n\}$ for evaluation of π , which we shall discuss in Sections 5 to 5.3 and 6 in details.

5. Determination of $\cos(p\pi/q)$

The infinite product formulas for $\sin(x)/x$ derived in Sections 3 and 4 involve terms of the form $\cos(p\pi/q)$ where p and q are positive integers. Evaluating $\cos(p\pi/q)$ accurately requires high-precision approximation of π . Depending upon the value of q , the solution of resulting from polynomial equation of degree q having a root $\cos(p\pi/q)$ is impossible in its coefficients due to Abel-Ruffini impossibility if $q \geq 5$. Although, the roots of the equation can be approximated by numerical analysis but the approximation error will impair the primary objective of the paper: the precise extraction of π . This limitation has historically affected both this specific solution and other applications involving higher-degree equations.

Coupled with the problem is the fact using cosine angle doubling formula

$$\cos(x) = \left[\frac{1}{2} \{1 + \cos(2x)\} \right]^{\frac{1}{2}} \quad (5.1)$$

can scale the angle up to $2^r(p\pi/q)$. If we try to determine the final value corresponding to the specific stage of angle doubling suited $\cos(p\pi/q)$, we are left uncertain of its precise value due to a lack of information between $\cos\left(\frac{2p\pi}{q}\right)$, and $\cos(2^{r-1}p\pi/q)$, where r is a positive integer.

Conceptually, we know that the cosine of an angle—being the ratio of the adjacent side (base) to the hypotenuse—becomes negative when the terminal side lies along the negative X-axis within the interval $\pi/2$ to $3\pi/2$. By applying formula (5.1) repeatedly, $\cos(p\pi/q)$ can be expressed in terms of $\cos(2^r p\pi/q)$. Since $\cos(x) > 0$ in $[0, \pi/2]$ and $\cos x < 0$ in $[\pi/2, 3\pi/2]$, the sign must be tracked during angle doubling.

To establish the initial positive threshold, *there exists a positive integer M such that the angle $p\pi/q$ doubled M times satisfies:*

$$2^M \left(\frac{p\pi}{q} \right) < \frac{\pi}{2}, \quad (5.2)$$

while doubling it at the next successive stage forces it across the quadrant boundary:

This implies that during the first M stages of angle doubling, the cosine values remain strictly positive, but at the $(M+1)$ -th stage of doubling, the value turns negative. The sequence of tracking signs (excluding the initial sign) can be written as:

$$+, \dots, (M \times) -, \dots$$

The exact boundary threshold parameter m is determined by setting the doubled angle equal to the quadrant boundary, $2^m \left(\frac{p\pi}{q} \right) = \frac{\pi}{2}$, yielding

$$m = \frac{\ln\left(\frac{q}{2p}\right)}{\ln(2)}. \quad (5.4)$$

If the value of m found out from the said equation be $M+c$, where M is an integer and $0 < c < 1$, then m will have integer value M and at stage, $M+1$, cosine of that angle will turn negative.

5.1 Nested Radical Construction and Sign Recurrence

We start from the known negative value $2^{M+1}(p\pi/q)$, and set $2^{M+1}(p\pi/q) = \text{negative quantity}$. Then we reflect it into the first quadrant: $-\cos(x_1) = \cos\{\pi - 2^{M+1}(p\pi/q)\}$.

Now express $(p\pi/q)$ as a nested radical by doubling the angle forward at each level and we see

$$\cos\left(\frac{p\pi}{q}\right) = \sqrt{\frac{1}{2} + \frac{1}{2} \cos\left(\frac{2p\pi}{q}\right)} = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \cos\left(\frac{4p\pi}{q}\right)}}.$$

Each nesting level corresponds to one angle doubling, from the inside out. Continue until the sign sequence repeats or terminates. There are three cases:

Case 1: Full Recurrence (Infinite Repeating Block)

If at some stage at angle x_j , the reflected angle returns to the original. The sign sequence repeats from the start, and the nested radical is purely periodic — the entire block repeats infinitely with no finite prefix.

$$\cos\left(\frac{p\pi}{q}\right) = \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} - \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} - \frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2}} \sqrt{\frac{1}{2} + i} \quad i$$

Case 2: Partial Recurrence (Fixed Prefix + Repeating Cycle)

If $x_i = 2^r(x_1)$ for some $r \leq M$, or a later x_j matches a prior scaled angle, the sign sequence has:

- A fixed initial segment
- Followed by an infinite repeating cycle.

$$+ + + + \dots + - + - \dots$$

The nested radical has a finite prefix, then enters an infinite loop.

Case 3: Termination at $\pi/2$

If during doubling, the angle reaches exactly $\pi/2$, i. e. $\pi/2 = 2^r(x_1)$, then $\cos \{2^r(x_1)\} = 0$. The nested radical terminates after k levels:

$$\cos\left(\frac{p\pi}{q}\right)=\sqrt{\frac{1}{2}+\frac{1}{2}}\sqrt{\frac{1}{2}+\dots\frac{1}{2}}\sqrt{\frac{1}{2}}$$

Illustration 1: Let $\cos(2\pi/15)$ be the term for determination of its value. In this case $p=2$, $q=15$ and using Equation (4.4), $m=1.906$. That makes $M=1$ and at stage $M+1$ (or at stage 2), the sign of cosine of that angle will change to negative. It will have sequence of signs

+ - .

Consider the case of $\cos(2\pi/81)$, here $p=2$, $q=81$, $m=\{\ln(81/4)\}/\ln(2)=4.339$. Therefore, $M=4$ and change of sign at stage $M+1=5$. Its sign sequence will be

$$+ + + + - i$$

Consider another case of $\cos(5\pi/121)$. In this example, $p=5$, $q=121$, $m=\{\ln(121/10)\}/\ln(2)=3.596$. Therefore, $M=3$ and change of sign at $M+1=4$. Its sign sequence will be

$$+ \quad + \quad + \quad - \quad i$$

Illustration 2: Let $\cos(2\pi/15)$ be the term for determination of its value. Its signs change at angle $8\pi/15$, when $\cos(8\pi/15)$ turns $-\cos(7\pi/15)$. Now at angle $14\pi/15$, $\cos(14\pi/15)$ again turns $-\cos(\pi/15)$. At angle $(2\pi/15)$, the recurrence starts as it was the starting point. Thus, $\cos(2\pi/15)$ will follow the sign sequence. See [10].

$$+ \quad - \quad - \quad + \quad \dots$$

In other words, it will follow signs

$$+ \quad - \quad - \quad + \quad - \quad - \quad + \quad - \quad - \quad + \quad - \quad - \quad + \dots$$

and $\cos(2\pi/15)$ will be expressed

$$\cos\left(\frac{2\pi}{15}\right)=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\dots}}}}$$

Now consider $\cos(\pi/25)$.

Its sign changes at:

$16\pi/25$: $\cos(16\pi/25)$ becomes $-\cos(9\pi/25)$,

$$18\pi/25: \cos(18\pi/25) \text{ becomes } -\cos(7\pi/25),$$
$$14\pi/25: \cos(14\pi/25) \text{ becomes } -\cos(11\pi/25),$$
$$22\pi/25: \cos(22\pi/25) \text{ becomes } -\cos(3\pi/25),$$
$$24\pi/25: \cos(24\pi/25) \text{ becomes } -\cos(\pi/25),$$

Finally, at $\pi/25$ (the starting point), the cycle recurs. Hence, the sign sequence for $\cos(\pi/25)$ is

$$+ + + - - - - + + + - \dots$$

and $\cos(\pi/25)$ is expressed

$$\cos\left(\frac{\pi}{25}\right) = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2}}}}}}}}}}}}}}}}}$$

See [10].

Illustration 3: Now consider $\cos(2\pi/81)$.

Its sign changes at:

$64\pi/81$: $\cos(64\pi/81)$ becomes $-\cos(17\pi/81)$,

$$68\pi/81: \cos(68\pi/81) \text{ becomes } -\cos(13\pi/81),$$
$$52\pi/81: \cos(52\pi/81) \text{ becomes } -\cos(29\pi/81),$$
$$58\pi/81: \cos(58\pi/81) \text{ becomes } -\cos(23\pi/81),$$

$46\pi/81$: $\cos(46\pi/81)$ becomes $-\cos(35\pi/81)$,

$70\pi/81$: $\cos(70\pi/81)$ becomes $-\cos(11\pi/81)$,

$44\pi/81$: $\cos(44\pi/81)$ becomes $-\cos(37\pi/81)$,

 $74\pi/81: \cos(74\pi/81) \text{ becomes } -\cos(7\pi/81),$
$$56\pi/81: \cos(56\pi/81) \text{ becomes } -\cos(25\pi/81),$$

$50\pi/81$: $\cos(50\pi/81)$ becomes $-\cos(31\pi/81)$,

$$62\pi/81: \cos(62\pi/81) \text{ becomes } -\cos(19\pi/81),$$
 $76\pi/81: \cos(76\pi/81) \text{ becomes } -\cos(5\pi/81),$
$$80\pi/81: \cos(80\pi/81) \text{ becomes } -\cos(\pi/81).$$

Finally, at $2\pi/81$ (the starting point), the cycle recurs. Hence, the sign sequence for $\cos(2\pi/81)$ is:

$$+ + + + - + - + - - - - + - - + + - - - + - + + + - + \dots$$

Illustration 4: There are some cosine trigonometric functions that do not exhibit recurrence but terminate after a fixed sign sequence. See [10]. Let us consider $\cos(\pi/24)$. Its sign changes at $16\pi/24$ and $\cos(16\pi/24)$ turns $-\cos(\pi/3)$, but $\cos(\pi/3)=1/2$. Its fixed sign sequence is + + + - and $\cos(\pi/24)$ is written:

$$\cos\left(\frac{\pi}{24}\right) = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{4}}}}} ,$$

(See [10].)

Illustration 5: Let us consider $\cos(\pi/32)$. At $16\pi/32$, $\cos(16\pi/32)$ becomes 0 and sign sequence + + + terminates. See [10]. $\cos(\pi/32)$ is written:

$$\cos\left(\frac{\pi}{32}\right) = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2}}}}}$$

5.2 Determination of $\sin(p\pi/q)$

- When the value of $\sin(p\pi/q)$ is to be determined by nested radicals, $\sin(p\pi/q)$ is written as $\cos(\pi/2 - p\pi/q)$ and methods as discussed in sections 5 to 5.1 are applied.
- Alternatively, angle doubling formula $\sin(p\pi/q) = \sqrt{1/2 - (1/2)\cos(2p\pi/q)}$ is applied. $\cos(2p\pi/q)$ so obtained in nested radical is successively doubled:

$$\sin\left(\frac{p\pi}{q}\right) = \sqrt{\frac{1}{2} - \left(\frac{1}{2}\right) \sqrt{\frac{1}{2} + \left(\frac{1}{2}\right) \sqrt{\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) \cos\left(\frac{8p\pi}{q}\right)}}$$

following the same procedure as detailed in sections 5 to 5.1.

6. Determination of $\sin\{\pi/(2K+1)^n\}$

Referring to section 4, it is observed, $\sin\{x/(2k+1)\}$ appears in the identity (4.2). Approximation of π in such cases is made at $x=\pi/2$, $\pi/4$ or $\pi/2^p$, where p is a positive integer. On substituting such values of x , the sine term takes the form $\sin[\pi/\{(2^p)(2k+1)\}]$. For ease in numerical calculations, it is better to assume $k=2^s$ and $p=1$, where s is a very large integer. This assumption results in transforming *sine* term to $\sin[\pi/\{2(2^{s+1}+1)\}]$. Using angle doubling formula,

$$\sin\left[\frac{\pi}{2(2^{s+1}+1)}\right] = \sqrt{\frac{1}{2} - \left(\frac{1}{2}\right) \cos\left\{\frac{\pi}{2^{s+1}+1}\right\}} .$$

On successively doubling the angle s times (or 2^s times),

$$\cos\left\{\frac{(2^s)\pi}{2^{s+1}+1}\right\} < \frac{\pi}{2}$$

and

$$\cos\left\{\frac{(2^{s+1})\pi}{2^{s+1}+1}\right\} > \frac{\pi}{2},$$

$$\cos\{2^{s+1}\pi/(2^{s+1}+1)\} > \pi/2,$$

when $\cos\{(2^{s+1}\pi)/(2^{s+1}+1)\} = -\cos\{\pi/(2^{s+1}+1)\}$ and the sign will change to negative. The sequence of signs will be - + + + ... continuing up to s times and repeating.

Nested radicals for $\sin[\pi/\{2(2^{s+1}+1)\}]$ will be expressed

$$\sin\left\{\frac{\pi}{2(2^{s+1}+1)}\right\} = \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \dots + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \dots + \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \dots}}}}}}}}}$$

Please note, signs ... used in nested radicals above means, “Nested radicals with positive signs continue for ‘s’ times, followed by a negative sign, and again positive signs for ‘s’ times, recursively.” If $p=P$, then beginning with negative sign, positive signs will continue up to $s+P-1$ times and then these will be followed by a negative sign and thereafter continue with positive signs up to s times. This recursive relation will continue.

Example: Let $s=6$, $k=2^6$, $p=2$ then $2k+1=2^7+1=129$.

[illegible]

Since $p=2$, its signs $- +$. Now $s=6$, there will be six positive signs or $++ ++$. Seventh sign will be negative or $-$. The sequence will be

— + + + + + + + — + + + + + + + — + + + + + + + — . .

Because of short width of this page, I preferred smaller s equal to 6 but in practice, s must be extremely large to have high precision even with one multiplicative term.

7. Family of Infinite Formulae of π Based on Sections 3.1 to 3.4

In Sections 3.1 to 3.4, various infinite product formulae for $\sin(x)/x$ have been derived. These formulae will be used to approximate the value of π . At $x = \pi/4$, the equation takes the following form:

$$\frac{\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} = \left\{ \frac{1}{3} + \left(\frac{2}{3}\right) \cos\left(\frac{\pi}{6}\right) \right\} \left\{ \frac{1}{3} + \left(\frac{2}{3}\right) \cos\left(\frac{\pi}{18}\right) \right\} \left\{ \frac{1}{3} + \left(\frac{2}{3}\right) \cos\left(\frac{\pi}{54}\right) \right\} \dots$$

Following the method given in sections 5 to 5.3,

$$\cos\left(\frac{\pi}{18}\right) = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \dots}}}}}}}}}}}}},$$

$$\cos\left(\frac{\pi}{54}\right) = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \dots}}}}}}}}}}}}},$$

where $\cos(2\pi 3^n)$ can be calculated and their values put in Equation (5.1) facilitates approximation of

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left\{ \frac{1}{3} + \left(\frac{2}{3} \right) \cos \left(\frac{2x}{3^n} \right) \right\}.$$

$$\frac{1}{\pi} = \left(\frac{1}{2^{\frac{3}{2}}} \right) \left\{ \frac{1}{3} + \left(\frac{2}{3} \right) \left(\frac{3^{\frac{1}{2}}}{2} \right) \right\} \left\{ \frac{1}{3} + \left(\frac{2}{3} \right) \cos \left(\frac{\pi}{18} \right) \right\} \left\{ \frac{1}{3} + \left(\frac{2}{3} \right) \cos \left(\frac{\pi}{54} \right) \right\} \dots$$

Using Equation (3.7) at $x = \pi/4$,

$$\frac{\sin \left(\frac{\pi}{4} \right)}{\frac{\pi}{4}} = \left\{ \frac{1}{5} + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{10} \right) + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{5} \right) \right\} \left\{ 1/5 + (2/5) \cos (\pi/50) \right. \\ \left. + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{25} \right) \right\} \left\{ \frac{1}{5} + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{250} \right) + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{125} \right) \right\} \dots$$

$\cos (\pi/5) = (1 + 5^{1/2})/4$, nested radical formula for $\cos (\pi/25)$ has already been derived in section 5.1. Values of $\cos(\pi/10)$, $\cos(\pi/50)$... are calculated using nested radicals

$$\cos \left(\frac{\pi}{10} \right) = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \dots}}} ,$$

$$\cos \left(\frac{\pi}{50} \right) = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} - \frac{1}{2} \dots}}}}}}}}}}}}}}}} \dots$$

Those yields

$$\frac{1}{\pi} = \left(\frac{1}{2^{\frac{3}{2}}} \right) \left\{ \frac{1}{5} + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{10} \right) + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{5} \right) \right\} \left\{ \frac{1}{5} + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{50} \right) + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{25} \right) \right\} \left\{ \frac{1}{5} + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{250} \right) + \left(\frac{2}{5} \right) \cos \left(\frac{\pi}{125} \right) \right\} \dots$$

Proceeding in this manner, value of π can be approximated

$$\frac{1}{\pi} = \left(\frac{1}{2^{\frac{3}{2}}} \right) \left\{ \frac{1}{7} + \left(\frac{2}{7} \right) \cos \left(\frac{\pi}{14} \right) + \left(\frac{2}{7} \right) \cos \left(\frac{3\pi}{14} \right) \right\} \left\{ \frac{1}{7} + \left(\frac{2}{7} \right) \cos \left(\frac{\pi}{98} \right) + \left(\frac{2}{7} \right) \cos \left(\frac{\pi}{49} \right) + \left(\frac{2}{7} \right) \cos \left(\frac{3\pi}{98} \right) \right\} \left\{ \frac{1}{7} + \left(\frac{2}{7} \right) \cos \left(\frac{\pi}{686} \right) + \left(\frac{2}{7} \right) \cos \left(\frac{\pi}{343} \right) + \left(\frac{2}{7} \right) \cos \left(\frac{3\pi}{686} \right) + \left(\frac{2}{7} \right) \cos \left(\frac{9\pi}{686} \right) \right\} \dots$$

Finally, at $x = \pi/4$, value of π in general is approximated

$$\frac{1}{\pi} = \left(\frac{1}{2^{\frac{3}{2}}} \right) \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} + \left(\frac{2}{2K+1} \right) \cos \left\{ \frac{\frac{2\pi}{4}}{(2K+1)^n} \right\} + \left(\frac{2}{2K+1} \right) \cos \left\{ \frac{\frac{4\pi}{4}}{(2K+1)^n} \right\} + \dots + \left(\frac{2}{2K+1} \right) \cos \left\{ \frac{\frac{2K\pi}{4}}{(2K+1)^n} \right\} \right]$$

8. Family of Infinite Formulae of π Based on Dirichlet Kernel Periodic Function

Referring to section 4, value of π can be approximated if value of $\sin \{ x/(2K+1)^n \}$ can be determined. In section 6, the method has been devised using nested radicals to determine this value. Equation (4.2) which is reproduced below,

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\left(\frac{1}{2K+1} \right) \frac{\sin\left\{ \frac{x}{(2K+1)^{n-1}} \right\}}{\sin\left\{ \frac{x}{(2K+1)^n} \right\}} \right]$$

will be used. For ease of calculations, only one multiplicative term is used.

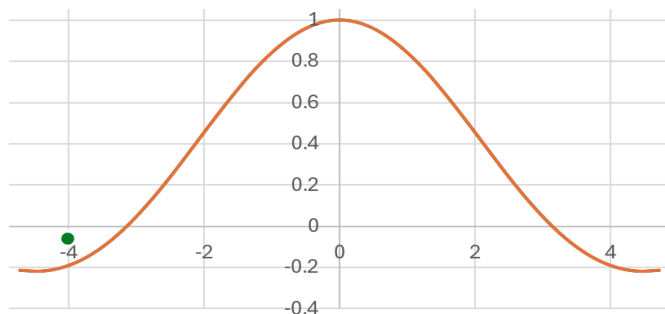


Figure 5: $\frac{\sin(x)}{x}$ and $\prod_{n=1}^{n=\infty} \left[\left(\frac{1}{2K+1} \right) \frac{\sin\left\{ \frac{x}{(2K+1)^{n-1}} \right\}}{\sin\left\{ \frac{x}{(2K+1)^n} \right\}} \right]$ curves

Two curves $\frac{\sin(x)}{x}$ and $\prod_{n=1}^{n=\infty} \left[\left(\frac{1}{2K+1} \right) \frac{\sin\left\{ \frac{x}{(2K+1)^{n-1}} \right\}}{\sin\left\{ \frac{x}{(2K+1)^n} \right\}} \right]$ are drawn taking their values along Y axis

and value of x along X axis. Value of K is considered, 2^{13} and only first multiplicative term is used. Both the curves as plotted, completely overlap confirming

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\left(\frac{1}{2K+1} \right) \frac{\sin\left\{ \frac{x}{(2K+1)^{n-1}} \right\}}{\sin\left\{ \frac{x}{(2K+1)^n} \right\}} \right].$$

Given below is a Table that displays how the percentage error varies with K .

Table 1 Variation of error with K

Value of K	Value of x	Multiplicative Terms Used	Calculated π	% Error
2^{13}	$\pi/6$	1	3.141592653	Correct up to 8 decimal points
2^{49}	$\pi/6$	1	3.141592654	Correct up to 31 decimal points

It is observed from the data in the table that as K in Equation 4.2 increases, the percentage error decreases with the use of only one multiplicative term.

9. Convergence and Rate of Convergence

To examining convergence of the infinite product of $\sin(x)/x$, the logarithm of both sides of the Equation (3.4) is taken and that transforms the expression into a series summation

$$\ln\left\{ \frac{\sin(x)}{x} \right\} = \sum_{n=1}^{n=\infty} \ln\left\{ \frac{1}{3} + \frac{2}{3} \cos\left(\frac{2x}{3^n} \right) \right\}.$$

Upon expanding $\cos(2x/3^n)$ using Maclaurin series,

$$\cos\left(\frac{2x}{3^n}\right) = 1 - \frac{1}{2!}\left(\frac{2x}{3^n}\right)^2 + \frac{1}{4!}\left(\frac{2x}{3^n}\right)^4 - \dots,$$

it takes the form

$$\ln\left\{\frac{1}{3} + \left(\frac{2}{3}\right)\cos\left(\frac{2x}{3^n}\right)\right\} \sim \ln\left[1 - \left(\frac{1}{3}\right)\left(\left(\frac{2x}{3^n}\right)^2\right)\right] \sim (4x^2)\left\{\frac{1}{3^{2n+1}}\right\}.$$

That shows that, n-th term of the series decays by an exponential factor of $1/3^{2n+1}$ and thus tends to zero. Generalising this to the division of any odd angle of the form $(2K+1)$, the nth logarithmic term decays by an exponential factor of $1/(2K+1)^{2n+1}$. For higher values of k, this decay is even faster indicating accelerated exponential convergence. Thus, the series exhibits exponential convergence, with a rate that increases with higher odd divisors of the angle.

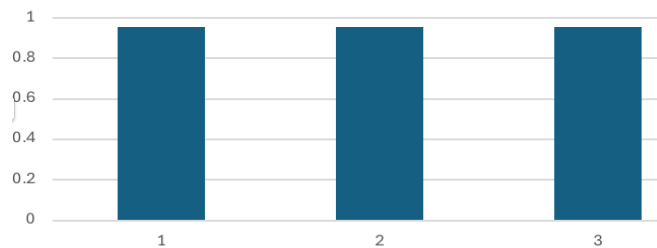


Figure 6 Displaying convergence with increase in multiplicative terms

In Figure 6, Bar 1 displays the calculated magnitude of $\sin(\pi/6)/(\pi/6)$ using only the first multiplicative term of Equation (4.2) with $K = 2^{12}$. Bar 2 shows the result using the first and second multiplicative terms, while Bar 3 includes the first, second, and third multiplicative terms. It is evident from the figure that the magnitudes of Bar 2 and Bar 3 are identical, and both closely match the actual value of $\sin(\pi/6)/(\pi/6)$. In other words, the product of the first, second, and third terms has already converged to the correct value of $\sin(\pi/6)/(\pi/6)$.

It is worth reiterating that even the first multiplicative term would suffice for convergence to $\sin(\pi/6)/(\pi/6)$, had a sufficiently large value of K been chosen. However, to demonstrate the convergence mechanism more clearly, K was intentionally chosen to be relatively smaller, specifically $K = 2^{12}$.

10. Comparison of the Method with Other Methods

10.1. Ramanujan-type and Other Series Methods for Approximating π

There is no doubt that Ramanujan-type and some other series methods exhibit extremely fast convergence yielding exact 8 or more digits of π required for high precision computation. However, they present severe arithmetic and digit-growth scaling issues if attempted via manual paper-and-pencil computation and lack the pedagogical transparency of nested radical structures. However, they require large factorials, modular functions, and algebraic numbers, which are not easily computed manually. They also involve *modular forms elliptic function and deep theorems beyond typical undergraduate level*. However, these are not based upon nested radical structure but require floating point computation of irrational constraints and factorials. These are known for elegant forms, and their compactness often hides deep complexity. These have superior speed and precision if one wants one hundreds or thousands of digits, but these methods are impractical for casual or as educational use.

10.2. Present Method

The method presented in this paper display exponential convergence depending upon chosen K. For instance, with $K = 2^{49}$, accuracy upto 31 decimal places is achieved using just one multiplicative t

term. The accuracy can be controlled by parameter K . Increasing K reduces the need for additional multiplicative terms. The method offers flexibility to trade complexity for precision. It has simple base formula involving trigonometric terms. It does not involve factorials or advanced number theoretic constant. The method can be understood and implemented by non-specialists. The method can be derived using basic complex number, trigonometry and binomial expansion and is conceptually accessible. The method involves trigonometric ratio $\cos\{x/(2K+1)^n\}$ which is expressible through nested radicals especially when $K=2^s$ with a predictable sign pattern. This makes it practical without complex exponentiation. It transforms using Dirichlet kernel into a compact product form. It expresses trigonometric behaviour through multiplication rather than summation.

11. Result and Conclusions

The π constant has a rich historical background and a fascinating nature that continues to intrigue mathematicians. Despite being calculated to billions of digits, its exact value remains elusive due to its transcendental character, which also renders the classical problem of squaring the circle impossible. While the transcendental nature of π was rigorously established arithmetically by Lindemann, it is proposed here that the curvedness of the circle is inherently and infinitely smooth—analogue to the curvature formed by a polygon with an infinite number of sides. This infinite smoothness means no finite representation or approximation can capture its true essence. This infinite non-repeating decimal reflects its transcendence.

In the derivation of angle-division identities involving $\sin(x)$, the expressions consistently contain a multiplier of the form $\sin\{x/(2K+1)\}$. This introduces a recursive structure where the next multiplier becomes $\sin\{x/(2K+1)^2\}$, and so on, iterating through $\sin\{x/(2K+1)^n\}$, with k as a positive integer. In the limit as $n \rightarrow \infty$, the expression $\sin(x)/\sin\{x/(2K+1)^n\}$ tends to $\sin(x)\{x/(2K+1)^n\}$ resulting in

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} + \left(\frac{2}{2K+1} \right) \cos \left\{ \frac{2x}{(2K+1)^n} \right\} + \left(\frac{2}{2K+1} \right) \cos \left\{ \frac{4x}{(2K+1)^n} \right\} + \dots + \left(\frac{2}{2K+1} \right) \cos \left\{ \frac{2jx}{(2K+1)^n} \right\} + \dots + \left(\frac{2}{2K+1} \right) \cos \left\{ \frac{2Kx}{(2K+1)^n} \right\} \right]$$

This reflects a scaled Dirichlet kernel structure that tends to unity and reveals convergence to an identity

$$\frac{\sin(x)}{x} = \prod_{n=1}^{n=\infty} \left[\frac{1}{2K+1} D_K \left\{ \frac{x}{(2K+1)^n} \right\} \right] = \prod_{n=1}^{n=\infty} \left[\left(\frac{1}{2K+1} \right) \frac{\sin \left\{ \frac{x}{(2K+1)^{n-1}} \right\}}{\sin \left\{ \frac{x}{(2K+1)^n} \right\}} \right]$$

Since terms like $\sin(px/q)$ emerge in these identities, their computation involves nested radicals using exclusively the number 2. The infinite product derived from this identity converges to π exponentially. For instance, using $K=2^{49}$, the value of π is computed correctly to 31 decimal places. Tabulated results from solved examples validate the precision and reliability of the identity derived.

Beyond its mathematical novelty, this work offers strong pedagogical value. By developing infinite product representations for π from first principles, the approach models a style of teaching that emphasises understanding over memorisation. Students can follow the gradual construction of identities using only elementary trigonometric knowledge, which promotes active learning and conceptual insight. The nested radical formulations further illustrate how sophisticated results can emerge from accessible ideas, making this work especially useful for both instructors and motivated learners.

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